EXERCISE 1	DETERMINATION OF MOMENT OF INERTIA OF PHYSICAL PENDULUM AND VERIFICATION OF PARALLEL AXIS THEOREM

Goals : to observe that the period of small oscillation of a physical pendulum depends on the moment of inertia, to determine the moment of inertia of chosen solid objects about the central axis.

Key words: center of mass, moment of inertia, moment of inertia of a rigid body, parallel axis (Steiner) theorem, physical pendulum, harmonic oscillations.

1. Introduction

The total of a physical quantity for a given system of *i* particles is defined as the sum of the quantity over all the particles constituting the system. Thus:

- (total) mass of a system: $M = \sum_i m_i$,
- (total) momentum of a system : $\vec{\mathbf{P}} = \sum_{\mathbf{i}} \vec{\mathbf{p}}_i$,
- (total) kinetic energy of a system: $K = \sum_{\mathbf{i}} K_i$.

For a motion of the system of particles the center of mass is the quantity of crucial importance. In the case of the translational motion, the system of particles can be replaced by a single particle of mass equal to the total mass *M* positioned at the center of mass. Newton's Second Law of Motion law is then simply given as:

$$\vec{\mathbf{F}}_{net} = M\vec{\mathbf{a}}_C , \tag{1}$$

with $\vec{\mathbf{F}}_{net}$ equal to the net force exerted on the system and $\vec{\mathbf{a}}_C$ - the center of mass acceleration. Thus the system of many particles can be treated as a single particle. For the discrete set of particles the position vector of the center of mass $\vec{\mathbf{R}}_c$, defined in the terms of the position vectors, is given as :

$$\vec{\mathbf{R}}_c = \frac{\sum_i m_i \vec{\mathbf{r}}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{\mathbf{r}}_i}{M} , \tag{2}$$

where $\vec{\mathbf{r}}_i$ is the position vector of a single *i*-particle. For a rigid body of continuous distribution of mass, the sum of masses in Eq.(2) must be generalized to integrals:

$$\vec{\mathbf{R}}_c = \frac{\int \vec{\mathbf{r}} dm}{m} = \frac{\int \rho \vec{\mathbf{r}} dV}{m} , \tag{3}$$

where dm is an elementary mass positioned at $\vec{\mathbf{r}}$ and ρ is the density of the body. The rotational analogue of Newton's Second Law is:

$$\vec{\mathbf{M}} = I\vec{\boldsymbol{\varepsilon}}\,, \tag{4}$$

with a torque $\vec{\mathbf{M}}$ - the rotational analogue of force, a moment of inertia I - the rotational analogue of mass and $\vec{\boldsymbol{\varepsilon}}$ - the angular acceleration. For a collection of many particles, arranged in a rigid configuration, the moment of inertia about an axis is defined as:

$$I = \sum_i m_i r_i^2\,, \tag{5}$$

where r_i is the distance of the *i-th* particle from the axis of rotation. When a rigid body is a continuous distribution of mass, the sum of masses and distances that defines the moment of inertia must be generalized to an integral. Dividing the body into small mass elements dm so that all points in a particular element are essentially at the same distance r from the axis of rotation one obtains an equation for the moment of inertia:

$$I = \int r^2 dm\,. \tag{6}$$

Steiner's law - parallel axis theorem

In this exercise the rotation of a rigid body about a fixed axis is studied. Let us assume that the moment of inertia of a rigid body about its central axis is given as I_0 (central axis is the axis which goes through the center of mass and is the axis of symmetry). Using the Steiner's law, it is possible to find the moment of inertia about any other axis parallel to the original one I_d, but displaced from it by a distance *d.* According to the law, the difference between the two moments of inertia is equal to the mass of the body multiplied by the square of the distance between these axes:

$$I_d = I_o + md^2 \tag{7}$$

Suppose a body of mass m is made to rotate about an axis z (see Fig. 1) passing through the body's center of gravity. The body has a moment of inertia I_0 with respect to this axis. The parallel axis theorem states that if the body is made to rotate instead about a new axis z' which is parallel to the first axis and displaced from it by a distance d, then the moment of inertia I_d with respect to the new axis is related to I_0 by the Eq. (7). A distance d is the perpendicular distance between the axes z and z'. Therefore Steiner's law is called also as "the parallel axis theorem".

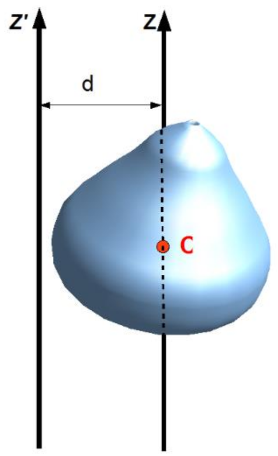

Fig. 1. A schematic picture being a graphical representation of Steiner's law.

2. Measurement method

The moment of inertia of a rigid body about the chosen axis can by determined experimentally using a physical pendulum. The oscillation period T for such a pendulum can be calculated for small angle approximation, where the application of Newton's Second Law leads to the harmonic oscillator equation, with period T given by the formula:

$$T=2\pi\sqrt{\frac{I}{mgd}}\ , \qquad (8)$$

where d is the distance between the axis of rotation and the center of mass of the pendulum and I is the moment of inertia about the axis of rotation. It can be rearranged to:

$$I_d = \frac{T^2 mgd}{4\pi^2}\ , \qquad (9)$$

which allows determination of the moment of inertia I_d about a given axis of rotation. The application of Steiner's law lets to calculate the moment of inertia about the central axis I_0:

$$I_0 = I_d - md^2. \qquad (10)$$

Such a procedure can be repeated for other axis of rotation, the calculated moments of inertia about the central axis for a given body should be the same within experimental uncertainty, which allows verifying Steiner's law. Its value can be also determined directly from geometrical dimensions of a given body (for simple shapes, such as a thin ring), which can be also used as a test of correctness of results.

3. Measurement tasks

A. Metal ring

a) Measure the mass m of the metal ring using a scale.

b) Measure several times the internal and external diameter of the ring (d and D, respectively).

c) Using the metal frame, hang the ring on the rod so that it creates a physical pendulum. Please notice that $d/2$ gives you the distance between the pivot point and the center of mass.

d) Measure the time t of 100 oscillations of the ring around the pivot point. Repeat this measurement several times in order to increase accuracy (at least 3 times).

e) Calculate the average time t for n=100 oscillations. Calculate the uncertainty related to t.

f) Calculate the average period of oscillations $T=t/n$ and its uncertainty ΔT. Assume Δn=1.

g) Calculate the experimental moment of inertia I_d and its uncertainty ΔI_d. Use the following equation for I_d:

$$I_d = \frac{T^2 mgd}{8\pi^2}$$

This is the moment of inertia related to the oscillations about the pivot point used in the experiment.

h) Use the parallel axis theorem to calculate the moment of inertia I_0 for the axis of rotation z passing through the disc's center of mass. To do this, use previously calculated I_d which has been determined for the axis of rotation parallel to z. Calculate the uncertainty ΔI_0. You will need the following equation for I_0 (parallel axis theorem):

$$I_0 = I_d - m\frac{d^2}{4}$$

i) Calculate the theoretical value of the moment of inertia for the central axis z according to equation:

$$I_{0,theoretical} = \frac{1}{8}m(d^2 + D^2)$$

Calculate also the uncertainty $\Delta I_{0,theoretical}$. Compare the theoretical value of the moment of inertia ($I_{0,theoretical}$) with the experimental one (I_0). If the parallel axis theorem holds, both values should be comparable in the limit of the experimental uncertainties ($\Delta I_{0,theoretical}$ and ΔI_0).

B. Perforated disc

a) Measure the mass *m* of the perforated disc using a scale.
b) Choose 3 pivot points located at different distances from the center of mass of the disc. For each of them, carefully measure the doubled distance *d* between the pivot point and the center of mass.
c) Choose a pivot point and hang the perforated disc on the metal rod. The distance between the pivot point and the center of mass is *d/2*. For this particular pivot point, measure time *t* for *n*=100 oscillations of the pendulum.
d) Repeat this measurement several times (at least 3 times).
e) Conduct the above measurements for the two other pivot points which were previously selected. At the end, you should have measurements of time t_i for at least 3 pivot points d_i. Remember to write down which time t_i corresponds to which distance d_i!
f) Calculate the average time *t* for *n*=100 oscillations for each distance *d*. Calculate the uncertainty related to *t*.
g) Calculate the average period of oscillations *T=t/n* and its uncertainty ΔT for each distance *d*. Assume Δn=1.
h) For each pivot point, calculate the experimental moment of inertia I_d and its uncertainty ΔI_d. Use the following equation for I_d:

$$I_d = \frac{T^2 mgd}{8\pi^2}$$

Remember to make these calculations for each distance *d*.

i) Verify the parallel axis theorem. In order to do this you have to calculate the moment of inertia I_0 for the rotation axis passing directly through the center of mass of the disc and parallel to the axis related to I_d. To do this, use the parallel axis theorem:

$$I_0 = I_d - m\frac{d^2}{4} = \frac{T^2 mgd}{8\pi^2} - m\frac{d^2}{4}$$

Remember to make these calculations for each distance *d*. If the parallel axis theorem holds, the obtained values should be comparable in the limit of experimental uncertainty. It is therefore necessary to calculate also the uncertainties ΔI_0.