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Exercise 53	OHM'S LAW FOR ALTERNATING CURRENT

Key words: DC current, AC current, RLC circuit, Ohm's law for alternating current.

1. Introduction

The basic advantage of alternating current is that it induces the magnetic field that surrounds the conductor, which makes possible the use of Faraday's law of induction, allowing simple control over the magnitude of an alternating potential difference with a device called a transformer. Thus the electric power can be transmitted through power lines efficiently at high voltage, which reduces the power lost as heat due to resistance of the wire, which strongly increases with the magnitude of current. Figure 1 shows a simple model of an AC generator. As the conducting loop is forced to rotate through the external magnetic field, a sinusoidally oscillating electromotive force (emf) ε is induced in the loop:

$$\varepsilon = \varepsilon_{\max} \sin(\omega t) \tag{1}$$

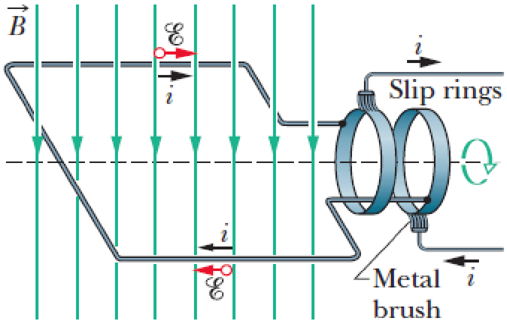

Fig. 1. The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field. In practice, the alternating emf induced in a coil of many turns of wire is made accessible by means of slip rings attached to the rotating loop. Each ring is connected to one end of the loop wire and is electrically connected to the rest of the generator circuit by a conducting brush against which the ring slips as the loop (and it) rotates.

The angular frequency ω of the emf is equal to the angular speed with which the loop rotates in the magnetic field, the phase of the emf is $\varepsilon\ t$, and the amplitude of the emf is ε_{max}. When the rotating loop is part of a closed conducting path, this emf produces (drives) a sinusoidal (alternating) current along the path with the same angular frequency ω, which then is called the driving angular frequency. We can write the current as:

$$I = I_{\max} \sin(\omega t - \phi) \tag{2}$$

in which I_{max} is the amplitude of the driven current. (The phase ωt - ϕ of the current is traditionally written with a minus sign.) We include a phase constant ϕ in Eq. 2 because the current I may not be in phase with the emf ε. (As you will see, the phase constant depends on the circuit to which the generator is connected.) We can also write the current I in terms of the driving frequency ω of the emf, by substituting $2\pi f$ for ω in Eq. 2:

$$I = I_{\max} \sin(2\pi f \cdot t - \phi) \tag{3}$$

The formulas for the charge (q) and potential/voltage (U) are similar, because the charge, potential, and current in both undamped LC circuits and damped RLC circuits (with small enough R) oscillate at angular frequency $\omega = 1/\sqrt{LC}$. Such oscillations are said to be free oscillations (free of any external emf), and the angular frequency ω is said to be the circuit's natural angular frequency.

$$U = U_{\max} \sin(\omega t - \phi) \quad \text{and} \quad q = q_0 \sin(\omega t - \phi) \tag{4}$$

When the external alternating emf of Eq. 1 is connected to an RLC circuit, the oscillations of charge, potential difference, and current are said to be driven oscillations or forced oscillations. These oscillations always occur at the driving angular frequency ω.
Later, we shall connect an external alternating emf device to a series RLC circuit (Fig. 2).

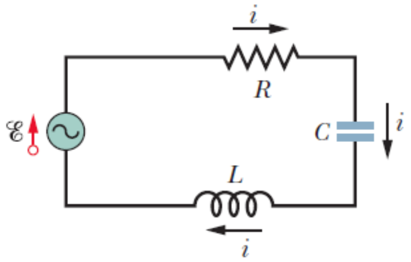

Fig. 2. A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and current are indicated here at only one instant.

We shall then find expressions for the amplitude I and phase constant ϕ of the sinusoidally oscillating current in terms of the amplitude ε_{max} and angular frequency ω of the external emf. First, let's consider three simpler circuits, each having an external emf and only one other circuit element: R, C, or L. We start with a resistive element (a purely resistive load).

2. A Resistive Load

Let us consider a circuit containing only a resistance element of the resistance R and an AC generator with the alternating emf of Eq. 1. By the loop rule, we have:

$$\varepsilon - U_R = 0. \tag{5}$$

With Eq. 1, this gives us:

$$U_R = \varepsilon_{\max} \sin(\omega t). \tag{6}$$

Because the amplitude U_{Rmax} of the alternating potential difference (or voltage) across the resistance is equal to the amplitude ε_{max} of the alternating emf, we can write this as:

$$U_R = U_{R\max} \sin(\omega t). \tag{7}$$

From the definition of resistance ($R=U/I$), we can now write the current I_R in the resistance:

$$I_R = \frac{U_R}{R} = \frac{U_{R\max}}{R}\sin(\omega t). \tag{8}$$

From Eq. 2, we can also write this current as:

$$I_R = I_{R\max}\sin(\omega t - \phi), \tag{9}$$

where I_{Rmax} is the amplitude of the current I_R in the resistance. Comparing Eqs. 8 and 9, we see that for a purely resistive load the phase constant $\phi = 0°$.

By comparing Eqs. 7 and 8, we see that the time-varying quantities U_R and I_R are both functions of *sin(ωt)* with $\phi = 0°$. Thus, these two quantities are in phase, which means that their corresponding maxima (and minima) occur at the same times. Figure 3, which is a plot of U_R(t) and I_R(t), illustrates this fact. Note that U_R and I_R do not decay here because the generator supplies energy to the circuit to make up for the energy dissipated in R.

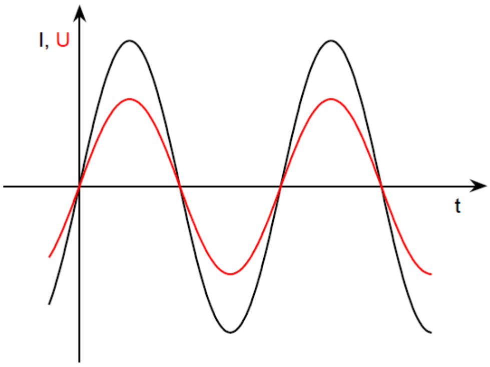

Fig. 3. The current I and the potential difference U across the resistor are plotted on the same graph, both versus time t.

3. A Capacitive Load

Let us consider a circuit containing only capacitance and a generator with the alternating emf of Eq. 1. Using the loop rule and proceeding as we did when we obtained Eq. 7, we find that the potential difference across the capacitor is:

$$U_C = U_{C\max}\sin(\omega t), \tag{10}$$

where U_{Cmax} is the amplitude of the alternating voltage across the capacitor. From the definition of capacitance we can also write:

$$q_C = C \cdot U_C = CU_{C\max}\sin(\omega t). \tag{11}$$

Our concern, however, is with the current rather than the charge. Thus, we differentiate Eq. 11 to find an expression for the current I_C:

$$I_C = \frac{dq_C}{dt} = \omega C U_C \cos(\omega t). \tag{12}$$

We now modify Eq. 12 in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_C, called **capacitive reactance** of a capacitor, defined as:

$$X_C = \frac{1}{\omega \cdot C} = \frac{1}{2\pi f \cdot C}. \qquad (13)$$

Its value depends not only on the capacitance but also on the driving angular frequency ω. Capacitive reactance X_C is in *ohms,* just as for resistance *R*.
Second, we replace $\cos(\omega t)$ in Eq. 12 with a phase-shifted sine:

$$\cos(\omega t) = \sin\left(\omega t + 90^0\right). \qquad (14)$$

With these two modifications, Eq. 12 becomes:

$$I_C = \frac{U_C}{X_C} \sin\left(\omega t + 90^0\right). \qquad (15)$$

From Eq. 3, we can also write the current I_C in the capacitor as:

$$I_C = I_{C\max} \sin(\omega t - \phi) = I_{C\max} \sin(2\pi f \cdot t - \phi), \qquad (16)$$

where I_{Cmax} is the amplitude of I_C. Comparing Eqs. 15 and 16, we see that for a purely capacitive load the phase constant ϕ for the current is -90°. Comparison of Eqs. 10 and 15, shows that the quantities U_C and I_C are 90°, $\pi/2$ rad, or one-quarter cycle, out of phase. Furthermore, we see that I_C leads U_C, which means that if you monitored the current I_C and the potential difference U_C, you would find that I_C reaches its maximum before U_C does, by one-quarter cycle (see Fig. 4).

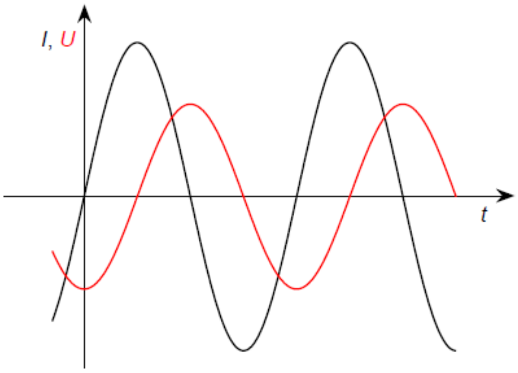

Fig. 4. The current in the capacitor leads the voltage by 90° (= π/2 rad).

4. An Inductive Load

Finally, let us consider a circuit containing only an inductance and a generator with the alternating emf of Eq. 1. Using the loop rule and proceeding as we did to obtain Eq. 7, we find that the potential difference across the inductance is:

$$U_L = U_{L\max} \sin(\omega t), \qquad (17)$$

where U_{Cmax} is the amplitude of the alternating voltage across the inductance. We can write the potential difference across an inductance L in which the current is changing at the rate dI_L/dt as:

$$U_L = L\frac{dI_L}{dt}. \tag{18}$$

If we combine Eqs. 17 and 18, we have:

$$\frac{dI_L}{dt} = \frac{U_L}{L}\sin(\omega t). \tag{19}$$

Our concern, however, is with the current, so we integrate Eq. 19:

$$I_L = \int dI_L = \frac{U_L}{L}\int \sin(\omega t)dt == -\left(\frac{U_L}{\omega L}\right)\cos(\omega t). \tag{20}$$

We now modify this equation in two ways. First, for the sake of symmetry of notation, we introduce the quantity X_L, called **inductive reactance** of an inductor, which is defined as:

$$X_L = \omega L = 2\pi f \cdot L. \tag{21}$$

The value of X_L depends on the driving angular frequency ω. The unit of the inductive reactance X_L is the ohm, just as it is for X_C and for R.
Second, we replace $-\cos(\omega t)$ in Eq. 20 with a phase-shifted sine:

$$-\cos(\omega t) = \sin(\omega t - 90^0). \tag{22}$$

With these two changes, Eq. 20 becomes:

$$I_L = \left(\frac{U_L}{X_L}\right)\sin(\omega t - 90^0). \tag{23}$$

From Eq. 3, we can also write the current I_L in the inductance as:

$$I_L = I_{L\max}\sin(\omega t - \phi) = I_{L\max}\sin(2\pi f \cdot t - \phi), \tag{24}$$

where I_{Lmax} is the amplitude of the current I_L. Comparing Eqs. 23 and 24, we see that for a purely inductive load the phase constant ϕ for the current is +90°. Comparison of Eqs. 17 and 24 shows that the quantities I_L and U_L are 90° out of phase. In this case, however, I_L lags U_L; that is, monitoring the current I_L and the potential difference U_L shows that I_L reaches its maximum value after U_L does, by one-quarter cycle (see Fig. 5).

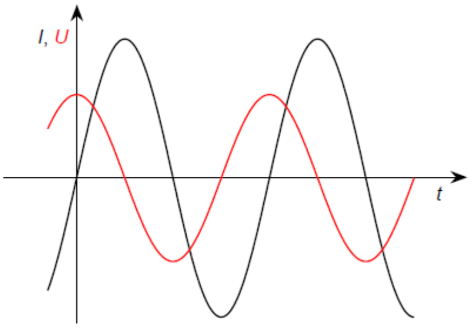

Fig. 5. The current in the inductor lags the voltage by 90° (= π/2 rad).

5. Ohm's law for RLC circuit

Ohm's law is an assertion that the current through a device is always directly proportional to the potential difference applied to the device. This assertion is correct only in certain situations; still, for historical reasons, the term "law" is used. The device in Fig. 6(left)—which is a 1000 Ω resistor—obeys Ohm's law. The device in Fig. 6(right)—which is called a p-n junction diode—does not. A conducting device is said to obey Ohm's law when the resistance of the device is independent of the magnitude and polarity of the applied potential difference.

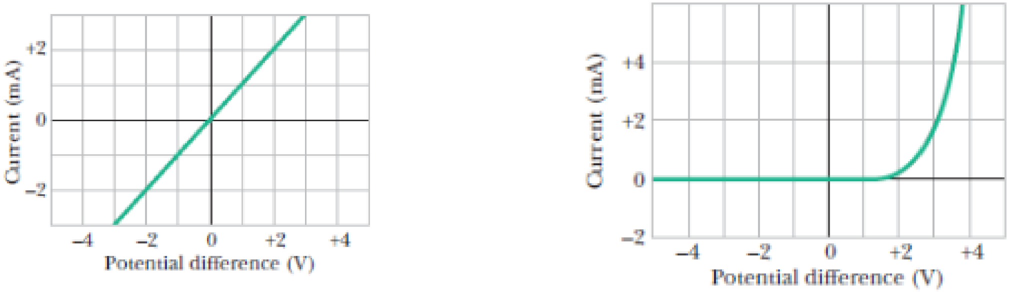

Fig. 6. (left) A plot of current I versus applied potential difference V when the device is a 1000 Ω resistor. (right) A plot when the device is a semiconducting p-n junction diode.

The familiar Ohm's law for DC circuits can only be used at AC if the load is purely resistive. Most AC circuits, however, contain series or parallel combinations of resistance, capacitance and inductance. This leads to the voltage and currents being out of phase and the load becoming complex. In circuits containing both inductors and capacitors, the voltage and current waveform will not be in phase, except at resonance. The general term for AC resistance is **impedance** and is given the symbol ***Z***.

In general, Ohm's law cannot be applied to alternating-current circuits since it does not consider the reactance which is always present in such circuits. However, by a modification of Ohm's law which does take into consideration the effect of reactance we obtain a general law which is applicable to AC circuits. Because the impedance, *Z*, represents the combined opposition of all the reactance's and resistances, this general law for AC is:

$$I_{\max} = \frac{\mathcal{E}_{\max}}{Z}, \tag{25}$$

where $Z = \sqrt{R^2 + (X_L - X_C)^2}$ for the RLC circuit.