EXERCISE 54	INVESTIGATION OF ELECTROMAGNETIC RESONANCE PHENOMENON

Goals: plotting of current-frequency characteristics of series RLC circuits, determination of resonance frequencies and quality factors of investigated circuits, determination of inductance of applied inductors.
Problems: alternate current, Faraday's law, RLC circuit, resonance phenomenon, quality factor.

1. Introduction

Resonance phenomenon is inherent for many systems under periodic external disturbance. It is revealed by an increase in the amplitude of system response for disturbance frequency equal to the natural frequency of the system. The most common is mechanical resonance, when the disturbance takes a form of external force, and the system response is a displacement from the equilibrium position, which in extremal cases may lead to destruction of the object. An example may be a crystal glass, which shatters when affected by high frequency sound. Resonance phenomenon is also the reason for the ban of organized marches on bridges, in order not to incite dangerous vibrations. The most famous example of the destructive strength of mechanical resonance is also related to a bridge in Tacoma, which fell apart under moderate wind blows (to watch http://youtu.be/3_AOvGOu3Dw).
Analogic phenomenon occurs in RLC circuits, i.e. circuits consisting of resistors, capacitors and inductors, including a source of AC voltage. The simplest series RLC circuit is shown in Fig. 1.

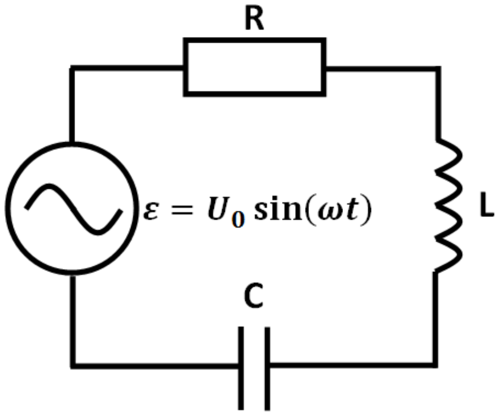

Fig. 1. Series RLC circuit.

In order to calculate the current *i(t)* in the circuit we must apply Kirchhoff's second law, which for the investigated system comes down to an equation

$$u_R(t) + u_C(t) = \varepsilon_L(t) + \varepsilon(t) \quad ,$$

meaning that a sum of voltage drops on the resistor $u_R(t)$ and the capacitor $u_C(t)$ is equal to the sum of electromotive force generated in the inductor $\varepsilon_L(t)$ and originating from the source $\varepsilon(t)$. Voltage drop on the resistor is given by Ohm's law

$$u_R(t) = R \cdot i(t) \ .$$

To calculate voltage drop on the capacitor we must use the relation between its capacitance *C,* momentary charge it accumulates *q(t)* and bias voltage $u_C(t)$

$$\mathrm{u_C}(t) = \frac{q(t)}{C} \quad .$$

The momentary charge accumulated on capacitor's plates depends on current in the circuit

$$\mathrm{q(t)} = \int \mathrm{i\,(t)d}t \quad .$$

Substituting this relation to the previous equation we get

$$\mathrm{u_C}(t) = \frac{\int \mathrm{i(t)d}t}{\mathrm{C}} \quad .$$

Electromotive force generated by AC current flowing through an inductor, according to the law of self-inductance, resulting from Faraday's law, is equal to

$$\varepsilon_\mathrm{L}(t) = -L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \quad ,$$

where *L* is the inductance of an inductor. On the other hand, temporal dependence of a voltage source is given by

$$\varepsilon(t) = U_0\sin(\omega t) \quad ,$$

where U_0 is a voltage amplitude, ω is an angular frequency of the source, connected to its frequency *f* by the relation

$$\omega = 2\pi f$$

After substitution to Kirchhoff's second law we obtain

$$R \cdot i(t) + \frac{\int \mathrm{id}t}{\mathrm{C}} = -L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + U_0\sin(\omega t) \quad .$$

In order to find the solution of that equation, i.e. to determine time dependence of current *i* flowing in the circuit, its terms must be ordered. In the first step terms dependent on current must be moved to the left side of the equation

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + R \cdot i(t) + \frac{\int \mathrm{i(t)d}t}{\mathrm{C}} = U_0\sin(\omega t) \quad .$$

Then it must be differentiated twice with respect to time *t*

$$L\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + R\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{\mathrm{i(t)}}{\mathrm{C}} = U_0\omega\cos(\omega t)$$

and divided by *L*

$$\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{\mathrm{CL}}\mathrm{i(t)} = \frac{U_0\omega}{L}\cos(\omega t) \quad .$$

The final equation is an inhomogeneous linear differential equation of the second order with constant coefficients. A solution of such equation is not very complicated, a suitable theory can be found in mathematical textbooks. Before the solution will be presented, a seemingly different problem will be considered, i.e. a harmonic movement of a mass attached to a spring under an external periodic force in damping medium. Dynamic equation of motion in such a case takes a form of

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + b\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega_0{}^2 x(t) = A\cos(\omega t)$$

where *x(t)* is a displacement from the equilibrium, *b* is a damping coefficient, ω_0 – a natural angular frequency of the system, *A* – an amplitude of an external force, ω – angular frequency of an external force. The comparison of both equations leads to the conclusion that they differ only in coefficients, thus the solution of one must also solve the other. For a damped harmonic oscillator with an external force, the solution is the dependence of a displacement on time given by periodic oscillations

$$x(t) = x_R \sin(\omega t + \varphi)$$

with certain phase φ, amplitude x_R having the following dependence on the angular frequency of the external force

$$x_R = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + (b\omega)^2}}.$$

Recalculating the relevant coefficients for the case of RLC circuit

$$b = \frac{R}{L} \;,$$

$$\omega_0^2 = \frac{1}{CL} \;,$$

$$A = \frac{U_0 \omega}{L} \;,$$

we obtain the current time dependence in the circuit

$$i(t) = I_0 \sin(\omega t + \varphi)$$

with the amplitude I_0 having the following dependence on the angular frequency of the source

$$I_0 = \frac{U_0}{\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}}.$$

The plot of that relation is shown in Fig. 2.

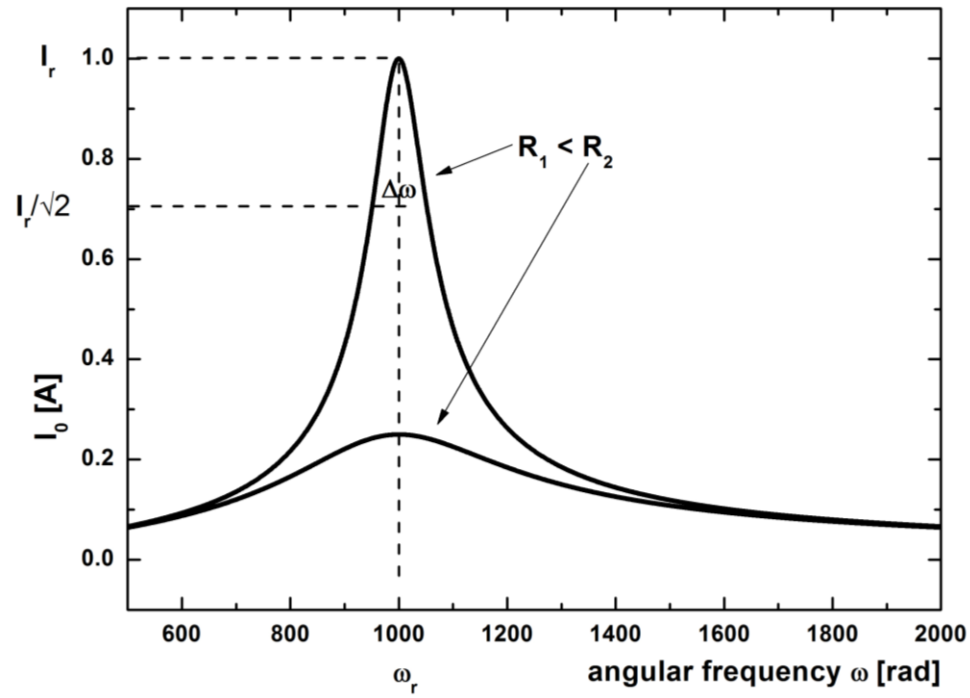

Fig. 2. Dependence of current amplitude (I_0) on angular frequency of voltage source (ω) in series RLC circuit.

The amplitude I_0 for fully determined angular frequency, called resonance frequency ω_r, reaches the maximum value of I_r. Analysis of the above equation leads to the conclusion that I_0 is largest when the denominator is smallest. The minimization of the denominator gives the relation

$$\omega L = \frac{1}{\omega C} \;.$$

Its solution allows the determination of the resonance angular frequency

$$\omega_{\mathrm{r}} = \frac{1}{\sqrt{CL}}$$

and the respective resonance frequency

$$f_{\mathrm{r}} = \frac{\omega_{\mathrm{r}}}{2\pi} = \frac{1}{2\pi\sqrt{CL}} \ .$$

It is worth noting that for the resonance frequency the current amplitude depends only on the resistance and the voltage amplitude of the source

$$I_{\mathrm{r}} = \frac{U_0}{R} \ .$$

Let us define the width of the resonance curve Δω as a distance on the angular frequency scale between points relating to the values of current $I = \frac{I_{\mathrm{r}}}{\sqrt{2}}$ (see Fig. 2.). Such quantity is dependent on the quality factor of the circuit Q, a dimensionless parameter describing the ratio of the energy stored in the system (in capacitors and inducers) during one period $T_{\mathrm{r}} = \frac{2\pi}{\omega_{\mathrm{r}}}$, to energy ΔE dissipated in resistors, given by

$$Q = 2\pi \frac{E}{\Delta E} \ .$$

The quality factor may be also expressed by its relations to RLC circuit parameters – it is inversely proportional to the resistance R and in the resonance conditions it can be expressed as

$$Q = \frac{\omega_{\mathrm{r}} L}{R} = \frac{1}{\omega_{\mathrm{r}} C R}$$

Or to the shape of the resonance curve – it is then expressed as the ratio of the angular resonance frequency ω_{r} to the width of the resonace curve Δω (defined above), it characterises the steepness of the resonance curve

$$Q = \frac{\omega_{\mathrm{r}}}{\Delta\omega} \ .$$

The easiest way to determine the quality factor experimentally is to measure the ratio of the voltage on the capacitor U_{C} (or on the inducer) at the resonance frequency to the voltage amplitude of the source U_0

$$Q = \frac{U_{\mathrm{C}}}{U_0} = \frac{U_{\mathrm{L}}}{U_0} \ .$$

As shown above, the response of the circuit containing resistors, capacitors and inducers connected in series, i.e. the current flowing through the circuit, strongly depends on the frequency of the source. For a fully determined frequency, called the resonance frequency, related to the angular frequency by $f_{\mathrm{r}} = \frac{\omega_{\mathrm{r}}}{2\pi}$, the current is the largest. By a careful choice of circuit elements it is possible to tune also the quality factor Q, which determines the width of the resonance curve Δω. Series RLC circuits can serve as a frequency filter, since it transmits (amplifies) only frequencies in a certain range, around the resonance frequency. The width of the transmission range (selectivity of the filter) depends on the quality of the circuit. A series RLC circuit can be used e.g. in TV or radio sets, were the tuning of system elements (usually capacitance of a capacitor) leads to the amplification of a chosen frequency, i.e. the choice of a given radio/TV channel.

2. Measurement method and measurement setup

A measurement setup used to obtain resonance curve is shown in Fig. 3. The setup includes:

- AC voltage source, with tunable amplitude and frequency
- Resitor, capacitor and inductor connected in series
- AC miliammeter, to measure current in the circuit
- AC voltmeter, to measure voltage on the capacitor

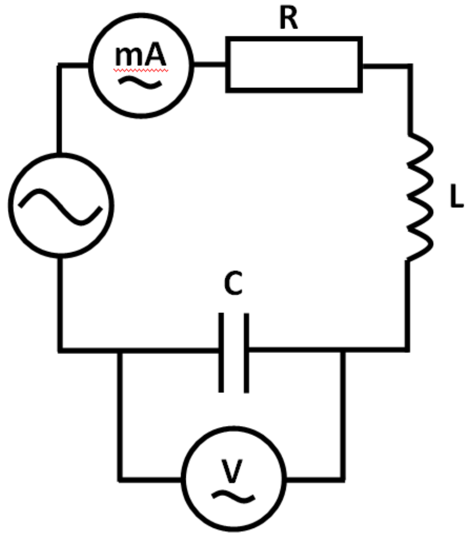

Fig. 3. Measurement circuit scheme.

Measurements are performed in two stages. First the resonance frequency of the system must be found by observing changes of current with frequency. Then, the final measurement must be planned in such a way to obtain properly the shape of resonance curve, meaning that the measurement of the current as a function of source frequency should start far from the resonance frequency and measurement points must be denser close to it. Additionally, to determine the quality factor of the circuit, voltage on the capacitor at the resonance must be measured. Repeat the measurements for other values of *R*, *L* and *C*, preferably keeping *C* constant.

3. Measurement tasks

a) Plot the characteristic of $I = I(f)$.
b) Determine the resonance frequency f_r and indicate it on the graph.
c) Include uncertainties for chosen measurement points, based on the uncertainties of measurement devices.
d) Calculate the capacity *C* from the equation

$$C = \frac{1}{(2\pi f_r)^2 L} \ . \tag{1}$$

e) Calculate the compound uncertainty of *C* based on the following equation

$$u_c(C) = \frac{1}{2\pi^2}\sqrt{\left[\frac{u(L)}{2 f_r^{\,2} L^2}\right]^2 + \left[\frac{u(f_r)}{f_r^{\,3} L}\right]^2} \ . \tag{2}$$

f) Calculate the quality factor of the circuit based on

$$Q = \frac{U_C}{U_0}. \tag{3}$$

g) Calculate its uncertainty

$$u_c(Q) = \sqrt{\left[\frac{u(U_C)}{U_0}\right]^2 + \left[\frac{U_C}{U_0^{\,2}} u(U_0)\right]^2} \ . \tag{4}$$

h) Estimate the value of *Q* from the characteristic of $I = I(f)$, according to the relation

$$Q = \frac{f_r}{\Delta f} \ , \tag{5}$$

where Δf is the width of a peak at square root of maximum amplitude. Compare it to the one determined in f). Estimate its uncertainty.
i) Repeat for all sets of RLC.

4. Questions:

- Describe in general the resonance phenomenon
- Explain analogy between electrical resonance in an RLC circuit and mechanical resonance for a damped harmonic oscillator
- Give Faraday's law, describe inductance and self-inductance phenomena
- Give Ohm's law for an AC series RLC circuit
- Describe the dependence of current on angular frequency in a series RLC circuit
- What is the dependence of the resonant frequency in a series RLC circuit on capacitance of a capacitor and inductance of an inductor?
- What is the physical meaning of quality factor for resonant circuit?
- What is the practical meaning of quality factor?
- Give some examples of practical application of electromagnetic resonance

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